

# Definition and Self-Adjointness of the Stochastic Airy Operator \*

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## Abstract

In this note, it is shown that the stochastic Airy operator, which is the “Schrödinger operator” on the half-line whose potential term consists of Gaussian white noise plus a linear term tending to  $+\infty$ , can naturally be defined as a generalized Sturm-Liouville operator and that it is self-adjoint and has purely discrete spectrum with probability one. Thus “stochastic Airy spectrum” of Ramírez, Rider and Virág is the spectrum of an operator in the ordinary sense of the word.

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## 1 Introduction.

In [2], Dumitriu and Edelman considered the following random matrix

$$H_n^\beta = \frac{1}{\sqrt{\beta}} \begin{bmatrix} g_1 & \chi_{(n-1)\beta} & 0 & \cdots & \cdots & 0 \\ \chi_{(n-1)\beta} & g_2 & \chi_{(n-2)\beta} & 0 & \cdots & 0 \\ & \cdots & \cdots & \cdots & \cdots & \cdots \\ & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \cdots & \cdots & \chi_{2\beta} & g_{n-1} & \chi_\beta \\ 0 & \cdots & \cdots & 0 & \chi_\beta & g_n \end{bmatrix}, \quad (1)$$

where  $\beta > 0$  is a real number and the random variables  $g_1, \dots, g_n$  obey the normal distribution  $N(0, 2)$ , whereas  $\chi_{j\beta}$  ( $j = 1, \dots, n-1$ ) obeys the  $\chi$ -distribution with parameter  $j\beta$ , all random variables being independent each other. Here by definition, a random variable  $X$  obeys the  $\chi$ -distribution if and only if  $X^2$  obeys the  $\chi^2$ -distribution. They called  $H_n^\beta$  the  $\beta$ -ensemble, and proved that the  $n$  eigenvalues of  $H_n^\beta$  have the joint probability density of the form

$$P_n^\beta(\lambda_1, \dots, \lambda_n) = \frac{1}{Z_n^\beta} e^{-\frac{\beta}{4} \sum_{k=1}^n \lambda_k^2} \prod_{j < k} |\lambda_j - \lambda_k|^\beta, \quad (2)$$

$Z_n^\beta$  being the normalizing constant. For the special values  $\beta = 1, 2, 4$ , (2) represents the joint probability density of the eigenvalues of the random matrix GOE, GUE and GSE respectively (see e.g. [7]). It was found out by Trotter [16] that these three well studied matrix ensembles can be transformed into the tridiagonal form (1), and in this form, the general  $\beta$ -ensemble interpolate and extrapolate those three ensembles.

Concerning the limiting distribution of the largest  $k$  eigenvalues  $\lambda_1^{(n)} \geq \dots \geq \lambda_k^{(n)}$  of  $H_n^\beta$ , Ramírez, Rider and Virág [11] proved that as  $n \rightarrow \infty$ , the joint distribution of  $\{n^{1/6}(2\sqrt{n} - \lambda_j^{(n)})\}_{j=1}^k$  converges to that of the smallest  $k$  eigenvalues  $\Lambda_0 \leq \dots \leq \Lambda_{k-1}$  of the “Schrödinger operator”

$$H = -\frac{d^2}{dt^2} + t + \frac{2}{\sqrt{\beta}} B'_\omega(t), \quad t \geq 0 \quad (3)$$

\*Dedicated to Professor Leonid Pastur on the occasion of his 75th birthday

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with Dirichlet boundary condition at  $t = 0$ , where  $\{B_\omega(t)\}_{t \geq 0}$  is the standard Brownian motion defined on a probability space  $(\Omega, \mathcal{F}, \mathbf{P})$ , and  $B'_\omega(t)$  is the formal derivative of its sample function.

Since sample functions of the Brownian motion are not differentiable, the expression (3) itself needs a justification. Ramírez et. al. consider  $H$  to be a random linear mapping from the function space

$$H_{loc}^1(\mathbf{R}_+) := \{f; f(\cdot) \text{ is absolutely continuous on } [0, \infty) \text{ and } f' \in L_{loc}^2(\mathbf{R}_+)\}$$

into the space of Schwartz distribution. They further define the eigenvalues and eigenfunctions of  $H$  to be pairs  $(\lambda, f) \in \mathbf{R} \times L^*$  such that  $Hf = \lambda f$  holds, both sides of which being interpreted as Schwartz distributions. Here the function space  $L^*$  is defined by

$$L^* = \{f \quad ; \quad f(\cdot) \text{ is absolutely continuous on } [0, \infty), f(0) = 0, \text{ and} \\ \int_0^\infty \{|f'(t)|^2 + (1+t)|f(t)|^2\} dt < \infty\} . \quad (4)$$

Actually, Schwartz distribution is irrelevant in formulating the eigenvalue problem for  $H$ , and it turns out that  $H$  is realized quite naturally as a self-adjoint operator in the Hilbert space  $L^2(\mathbf{R}_+)$ . Namely we have the following

**Theorem 1** *For each  $\omega$ ,  $H$  can be realized as a closed symmetric operator in  $L^2(\mathbf{R}_+)$ , which is self-adjoint with probability one and has purely discrete spectrum.*

Thus what Ramírez et.al. called “stochastic Airy spectrum” is the spectrum of a self-adjoint operator in the ordinary sense of the word. We shall prove Theorem 1 as an immediate corollary of the following more general result.

**Theorem 2** *Let  $p(t) \geq 0$  be a real valued continuous function on  $\mathbf{R}_+$  such that  $\liminf_{t \rightarrow \infty} p(t)/t^\alpha > 0$  for some  $\alpha > 0$ . Further let  $\{X_\omega(t)\}$  be a fractional Brownian motion of Hurst parameter  $h \in (0, 1)$  defined on a probability space  $(\Omega, \mathcal{F}, \mathbf{P})$ . Then under the Dirichlet boundary condition at  $t = 0$ ,*

$$H := -\frac{d^2}{dt^2} + p(t) + cX'_\omega(t) , \quad t \geq 0 \quad (5)$$

*can be realized as a closed symmetric operator in the Hilbert space  $L^2(\mathbf{R}_+)$ , which is self-adjoint and has purely discrete spectrum with probability one.*

Here a fractional Brownian motion  $\{X_\omega(t)\}$  of Hurst parameter  $h \in (0, 1)$  is a stochastic process satisfying the following conditions:

- (i)  $X_\omega(0) = 0$  and  $X_\omega(t)$  is continuous with respect to  $t \geq 0$  for all  $\omega \in \Omega$ ;
- (ii)  $\{X_\omega(t)\}$  is a centered Gaussian process;
- (iii)  $\mathbf{E}[X(t)X(s)] = \frac{1}{2}(t^{2h} + s^{2h} - |t - s|^{2h})$ .

See Nourdin [10] for details.

Since the standard Brownian motion is the fractional Brownian motion of Hurst parameter  $h = 1/2$ , Theorem 1 follows from Theorem 2 by letting  $p(t) = t$  and  $c = 2/\sqrt{\beta}$ .

In the following section, we give a definition of the operator  $H$  as a closed symmetric operator. The argument there is purely deterministic. In §3, we consider a random quadratic form

$$\begin{aligned} \mathcal{E}_\omega(u, v) &= \int_0^\infty \{u'(t)\overline{v'(t)} + (p(t) + ca'_\omega(t))u(t)\overline{v(t)}\} dt \\ &\quad - \int_0^\infty c(X_\omega(t) - a_\omega(t))(u(t)\overline{v(t)})' dt , \end{aligned} \quad (6)$$

which is defined for  $u, v \in L$ , where

$$L := \{f \quad ; \quad f(\cdot) \text{ is absolutely continuous on } [0, \infty), f(0) = 0, \text{ and} \\ \int_0^\infty \{|f'(t)|^2 + (1 + p(t))|f(t)|^2\} dt < \infty\} . \quad (7)$$

Here we have set, as in [11],

$$a_\omega(t) = \int_t^{t+1} X_\omega(s) ds . \quad (8)$$

It will then be shown that for  $\mathbf{P}$ -almost all  $\omega$ ,  $\mathcal{E}_\omega$  is closed, lower semi-bounded and completely continuous. Hence by general theory, there corresponds a self-adjoint operator  $A_\omega$  which is lower semi-bounded and has purely discrete spectrum. To complete the proof of Theorem 2, we then verify that this  $A_\omega$  is a self-adjoint extension of the symmetric operator defined in §2 and that this symmetric operator is of limit point type at  $+\infty$ .

## 2 Definition of $H$ as a symmetric operator.

Let  $p(t)$  and  $Q(t)$  be real valued continuous functions on  $[0, \infty)$  with  $Q(0) = 0$ , and consider the expression

$$H = H(p, Q) = -\frac{d^2}{dt^2} + p(t) + Q'(t) , \quad (9)$$

where  $Q'(t)$  is the formal derivative of  $Q(t)$ .

**Definition 1** ([8]) *A complex valued function  $u(t)$  belongs to the space  $\mathcal{C} = \mathcal{C}(p, Q)$  if and only if  $u$  is absolutely continuous on  $[0, \infty)$  and if there are  $\alpha \in \mathbf{C}$  and  $v(\cdot) \in L_{loc}^1(\mathbf{R}_+)$  such that*

$$u'(t) = \alpha + Q(t)u(t) + \int_0^t \{p(y)u(y) - Q(y)u'(y) - v(y)\} dy . \quad (10)$$

It is clear that  $v$  is uniquely determined from  $u$  as an element of  $L_{loc}^1(\mathbf{R}_+)$ . We define  $Hu = H(p, Q)u = v$  for each  $u \in \mathcal{C}$ .

If we define, following Savchuk and Shkalikov [14], the “quasi-derivative” of  $u$  by  $u^{[1]}(t) = u'(t) - Q(t)u(t)$ , then it is easily seen that  $u \in \mathcal{C}$  is equivalent to saying that  $u(t)$  and  $u^{[1]}(t)$  are absolutely continuous and satisfy

$$v(t) := -(u^{[1]}(t))' + (p(t) - Q^2(t))u(t) - Q(t)u^{[1]}(t) \in L_{loc}^1(\mathbf{R}_+) . \quad (11)$$

In fact this  $v$  coincides with  $Hu$  in Definition 1.

The equation  $Hu = \lambda u$  for  $\lambda \in \mathbf{C}$  is interpreted as the system of integral equations

$$\begin{aligned} u(t) &= u(0) + \int_0^t u'(s) ds ; \\ u'(t) &= u'(0) + u(0)(Q(t) + P(t) - \lambda t) \\ &\quad + \int_0^t \{Q(t) - Q(y) - \lambda(t - y) + P(t) - P(y)\} u'(y) dy , \end{aligned} \quad (12)$$

where  $P(t) := \int_0^t p(s) ds$ , or of differential equations

$$\frac{d}{dt} \begin{bmatrix} u(t) \\ u^{[1]}(t) \end{bmatrix} = \begin{bmatrix} Q(t) & 1 \\ \lambda - Q^2(t) - p(t) & -Q(t) \end{bmatrix} \begin{bmatrix} u(t) \\ u^{[1]}(t) \end{bmatrix} , \quad (13)$$

with a given initial condition, both (12) and (13) being uniquely solvable. Moreover, the Green’s formula is valid in the sense that for any  $u_1, u_2 \in \mathcal{C}$  and  $0 \leq a < b < \infty$ , one has

$$\int_a^b \{(Hu_1)(t)u_2(t) - u_1(t)(Hu_2)(t)\} dt = [u_1, u_2](t) \Big|_{t=a}^{t=b} , \quad (14)$$

where

$$[u_1, u_2](t) := u_1(t)u_2'(t) - u_1'(t)u_2(t) = u_1(t)u_2^{[1]}(t) - u_1^{[1]}(t)u_2(t) . \quad (15)$$

With these interpretations, it can be verified that  $H$  can be treated quite similarly as the classical Sturm-Liouville operator. For example, if we define two spaces

$$\mathcal{D}_0 := \{u \in \mathcal{C} \cap L^2(\mathbf{R}_+); u(0) = 0, \quad Hu \in L^2(\mathbf{R}_+)\} \quad (16)$$

and

$$\mathcal{D}_0^S := \{u \in \mathcal{D}_0; \lim_{t \rightarrow \infty} [u, v](t) = 0 \text{ for any } v \in \mathcal{D}_0\} , \quad (17)$$

and if we let

$$H_0 = H|_{\mathcal{D}_0}; \quad H_0^S = H|_{\mathcal{D}_0^S} , \quad (18)$$

then we can prove that  $H_0^* = H_0^S$  and  $(H_0^S)^* = H_0$  hold in exactly the same way as the proof of Theorem 10.11 of [15]. (From that argument, it follows in particular that  $\mathcal{D}_0^S$  is dense in  $L^2(\mathbf{R}_+)$ .) Thus  $H_0^S$  is a closed symmetric operator. Moreover, Weyl-Titchmarsh theory is valid also for  $H$ , and it holds that  $H_0^S$  is self-adjoint if and only if  $H$  is of limit point type at  $+\infty$ , which is true if and only if for some  $\lambda \in \mathbf{C}$ , the equation  $Hu = \lambda u$  has a solution which is *not* square integrable near  $+\infty$ .

Taking  $Q(t) = cX_\omega(t)$  in the definition of  $H_0^S$  above, we can realize the stochastic Airy operator as a closed symmetric operator in  $L^2(\mathbf{R}_+)$ .

We shall denote by  $H_0(\omega)$  and  $H_0^S(\omega)$  the operators defined by (18) with this choice of  $Q(t)$ . Also we denote by  $H(\omega)$  the expression (9) itself with the same choice of  $Q(t)$ .

**Remark 1** *As is pointed out by Edelman and Sutton [4] and Bloemendal [1], we can transform  $H$  in (9) to a classical Sturm-Liouville operator*

$$Lf = -\frac{d^2}{dt^2}f + pf - 2Q\frac{d}{dt}f - Q^2f$$

by letting  $u = f\phi$ , with  $\phi(t) = \exp\left(\int_0^t Q(x)dx\right)$ . This makes it much clearer that Weyl-Titchmarsh theory is still valid for our generalized Sturm-Liouville operator  $H$ .

For a recent systematic treatment of generalized Sturm-Liouville operator including those described in §2, see for example [3].

### 3 Almost sure self-adjointness of the stochastic Airy operator.

In this section, we construct a self-adjoint operator  $A_\omega$  which is associated with the quadratic form  $\mathcal{E}_\omega$  defined in (6), and show that  $A_\omega = H_0^S(\omega)$ . All necessary assertions concerning  $\mathcal{E}_\omega$  are deduced from the lemma below.

**Lemma 1** *If  $\{X_\omega(t)\}$  is the fractional Brownian motion with Hurst parameter  $h \in (0, 1)$ , then with probability one,  $a'_\omega(t) = \mathcal{O}(\sqrt{\log t})$  and  $X_\omega(t) - a_\omega(t) = \mathcal{O}(\sqrt{\log t})$  as  $t \rightarrow \infty$ , where  $a_\omega(t)$  is defined by (8).*

**Proof.** Let  $n \leq t \leq n+1$ . We then have

$$|a'_\omega(t)| = |X_\omega(t+1) - X_\omega(t)| \leq |X_\omega(t+1) - X_\omega(n+1)| + |X_\omega(n+1) - X_\omega(n)| + |X_\omega(t) - X_\omega(n)| ,$$

and hence

$$\sup_{n \leq t \leq n+1} |a'_\omega(t)| \leq 2 \sup_{0 \leq s \leq 1} |X_\omega(n+s) - X_\omega(n)| + \sup_{0 \leq s \leq 1} |X_\omega(n+1+s) - X_\omega(n+1)| .$$

Since the process  $\{X_\omega(t+T) - X_\omega(T)\}_{t \geq 0}$  has the same law as  $\{X_\omega(t)\}_{t \geq 0}$  for each  $T \geq 0$ , we see that for  $b > 0$  and  $n \geq 1$ ,

$$\begin{aligned} & \mathbf{P}\left(\sup_{n \leq t \leq n+1} |a'_\omega(t)| \geq b\sqrt{\log n}\right) \\ & \leq \mathbf{P}\left(\sup_{0 \leq s \leq 1} |X_\omega(n+s) - X_\omega(n)| \geq \frac{b}{3}\sqrt{\log n}\right) + \mathbf{P}\left(\sup_{0 \leq s \leq 1} |X_\omega(n+1+s) - X_\omega(n+1)| \geq \frac{b}{3}\sqrt{\log n}\right) \\ & = 2\mathbf{P}\left(\sup_{0 \leq s \leq 1} |X_\omega(s)| \geq \frac{b}{3}\sqrt{\log n}\right). \end{aligned}$$

Now it is known (see Theorem 4.1 of [10]) that for the fractional Brownian motion with Hurst parameter  $h \in (0, 1)$  the estimate

$$\lim_{x \rightarrow \infty} x^{-2} \log \mathbf{P}\left(\sup_{0 \leq s \leq 1} X_\omega(s) \geq x\right) = -\frac{1}{2}$$

holds. Hence for any  $\varepsilon \in (0, 1)$ , we can choose a  $K_\varepsilon > 0$  such that

$$\mathbf{P}\left(\sup_{0 \leq s \leq 1} |X_\omega(t)| \geq x\right) \leq K_\varepsilon e^{-(1-\varepsilon)x^2/2}.$$

Hence for  $b$  greater than  $\sqrt{2/(1-\varepsilon)}$ , we have

$$\begin{aligned} \sum_{n \geq 1} \mathbf{P}\left(\sup_{n \leq t \leq n+1} |a'_\omega(t)| \geq b\sqrt{\log n}\right) & \leq \sum_{n \geq 1} 2K_\varepsilon \exp\left[-\frac{1-\varepsilon}{2}b^2 \log n\right] \\ & = 2K_\varepsilon \sum_{n \geq 1} n^{-(1-\varepsilon)b^2/2} < \infty, \end{aligned}$$

and hence by Borel-Cantelli lemma, for  $\mathbf{P}$ -almost all  $\omega \in \Omega$ , we can choose a constant  $C_\omega$  so that

$$\sup_{n \leq t \leq n+1} |a'_\omega(t)| \leq C_\omega \sqrt{\log n} \quad (n \geq 1).$$

As to the assertion for  $X_\omega(t) - a_\omega(t)$ , we first note

$$\sup_{n \leq t \leq n+1} |X_\omega(t) - a_\omega(t)| = \sup_{n \leq t \leq n+1} \left| \int_0^1 (X_\omega(t+s) - X_\omega(t)) ds \right| \leq \sup_{n \leq t \leq n+1} \sup_{0 \leq s \leq 1} |X_\omega(t+s) - X_\omega(t)|$$

and

$$\begin{aligned} & |X_\omega(t+s) - X_\omega(t)| \\ & \leq \begin{cases} |X_\omega(t) - X_\omega(n)| + |X_\omega(t+s) - X_\omega(n)| & (0 \leq s \leq n+1-t) \\ |X_\omega(t+s) - X_\omega(n+1)| + |X_\omega(n+1) - X_\omega(n)| + |X_\omega(t) - X_\omega(n)| & (n+1-t \leq s \leq 1) \end{cases} \end{aligned}$$

Then as in the case of  $a'_\omega(t)$ , we again have

$$\sup_{n \leq t \leq n+1} |X_\omega(t) - a_\omega(t)| \leq 2 \sup_{0 \leq s \leq 1} |X_\omega(n+s) - X_\omega(n)| + \sup_{0 \leq s \leq 1} |X_\omega(n+1+s) - X_\omega(n+1)|,$$

and the rest of the proof is the same as the preceding argument.

The following proposition summarizes all the properties of  $\mathcal{E}_\omega$  which we need.

**Proposition 1** *Fix an  $\omega$  for which the conclusion of Lemma 1 holds. Then under the assumption of Theorem 2, we have that*

- (i)  $\mathcal{E}_\omega(u, v)$  is well defined for all  $u, v \in L$ ;
- (ii)  $\mathcal{E}_\omega$  is lower semi-bounded;
- (iii)  $\mathcal{E}_\omega$  is closed;
- (iv)  $\mathcal{E}_\omega$  is completely continuous in the sense that any  $L^2$ -bounded sequence  $\{u_n\}$  in  $L$  such that  $\sup_n \mathcal{E}_\omega(u_n, u_n) < \infty$  contains an  $L^2$ -convergent subsequence.

**Proof.** By Lemma 1, we have  $p(t) + ca'_\omega(t) = \mathcal{O}(1 + p(t))$  and  $X_\omega(t) - a_\omega(t) = o(\sqrt{1 + p(t)})$  as  $t \rightarrow \infty$ . In particular for any  $\varepsilon \in (0, 1/c)$ , one can choose a constant  $C_1 = C_1(\omega, \varepsilon)$  so that  $|X_\omega(t) - a_\omega(t)| \leq \varepsilon \sqrt{C_1 + p(t)}$  holds for all  $t \geq 0$ . This together with Schwarz inequality shows that the integrals in (6) are absolutely convergent for all  $u, v \in L$ , and hence  $\mathcal{E}_\omega$  is well defined on  $L$ . To verify assertion (ii), we write

$$\begin{aligned} \mathcal{E}_\omega(u, u) &= \int_0^\infty |u'(t)|^2 dt + \int_0^\infty (p(t) + ca'_\omega(t)) |u(t)|^2 dt \\ &\quad - c \int_0^\infty (X_\omega(t) - a_\omega(t)) \left( u(t) \overline{u'(t)} + u'(t) \overline{u(t)} \right) dt \end{aligned} \quad (19)$$

for  $u \in L$ . Noting  $2|xy| \leq |x|^2 + |y|^2$ , we see that the third integral on the right hand side of (19) is bounded in absolute value by

$$2\varepsilon c \int_0^\infty \sqrt{C_1 + p(t)} |u(t)| |u'(t)| dt \leq \varepsilon c \left\{ \int_0^\infty (C_1 + p(t)) |u(t)|^2 dt + \int_0^\infty |u'(t)|^2 dt \right\},$$

which is finite. Hence we have

$$\begin{aligned} \mathcal{E}_\omega(u, u) &\geq (1 - \varepsilon c) \int_0^\infty |u'(t)|^2 dt - \varepsilon c C_1 \int_0^\infty |u'(t)|^2 dt \\ &\quad + \int_0^\infty \{(1 - \varepsilon c)p(t) + ca'_\omega(t)\} |u(t)|^2 dt \end{aligned} \quad (20)$$

If we further take a  $\delta \in (0, 1 - \varepsilon c)$ , then there is a constant  $C_2 = C_2(\omega, p(\cdot), c, \varepsilon, \delta)$  such that

$$\inf_{t \geq 0} \{(1 - \varepsilon c - \delta)p(t) + ca'_\omega(t)\} \geq -C_2. \quad (21)$$

Hence letting  $C = \varepsilon c C_1 + C_2$ , we obtain the lower bound

$$\begin{aligned} \mathcal{E}_\omega(u, u) &\geq (1 - \varepsilon c) \int_0^\infty |u'(t)|^2 dt + \delta \int_0^\infty p(t) |u(t)|^2 dt - C \int_0^\infty |u(t)|^2 dt \\ &\geq -C \int_0^\infty |u(t)|^2 dt = -C(u, u), \end{aligned} \quad (22)$$

which proves the assertion (ii).

Now fix a constant  $\gamma > C$  and define the norm  $\|\cdot\|_\gamma$  on  $L$  by  $\|u\|_\gamma^2 = \mathcal{E}_\omega(u, u) + \gamma(u, u)$ . Then by (22),

$$\|u\|_\gamma^2 \geq (1 - \varepsilon c) \int_0^\infty |u'(t)|^2 dt + \delta \int_0^\infty p(t) |u(t)|^2 dt + (\gamma - C) \int_0^\infty |u(t)|^2 dt \quad (23)$$

holds for  $u \in L$ . To show that  $L$  is complete under the norm  $\|\cdot\|_\gamma$ , let  $\{u_n\} \subset L$  be a Cauchy sequence with respect to this norm. For this sequence, we have from (23)

$$\lim_{n, m \rightarrow \infty} \|u_n - u_m\|_2 = 0, \quad (24)$$

$$\lim_{n, m \rightarrow \infty} \|u'_n - u'_m\|_2 = 0, \quad (25)$$

and

$$\lim_{n,m \rightarrow \infty} \int_0^\infty p(t) |u_n(t) - u_m(t)|^2 dt = 0. \quad (26)$$

By (24) and (25), there are  $u, v \in L^2(\mathbf{R}_+)$  such that  $\|u_n - u\|_2 \rightarrow 0$  and  $\|u'_n - v\|_2 \rightarrow 0$  as  $n \rightarrow \infty$ . In particular we have  $\int_0^x |u'_n(t) - v(t)| dt \rightarrow 0$  for any  $x > 0$ . Hence we can let  $n \rightarrow \infty$  in the equality  $u_n(x) = \int_0^x u'_n(t) dt$  and we see that  $\{u_n(x)\}_n$  converges pointwise to a continuous function which must coincide almost everywhere with  $u(t)$ , namely  $u(x) = \int_0^x v(t) dt$  for almost every  $x$ . Thus  $u(t)$  is absolutely continuous,  $u(0) = 0$ , and  $u'(t) = v(t) \in L^2(\mathbf{R}_+)$ . By (26) and Fatou's lemma we get

$$\int_0^\infty p(t) |u(t)|^2 dt \leq \liminf_{n \rightarrow \infty} \int_0^\infty p(t) |u_n(t)|^2 dt < \infty,$$

so that  $u \in L$ , and

$$\lim_{n \rightarrow \infty} \int_0^\infty p(t) |u_n(t) - u(t)|^2 dt = 0.$$

Since we can choose a constant  $K_\omega$  such that

$$\mathcal{E}_\omega(u, u) \leq K_\omega \left\{ \int_0^\infty |u(t)|^2 dt + \int_0^\infty |u'(t)|^2 dt + \int_0^\infty p(t) |u(t)|^2 dt \right\}$$

in the same way as we proved (23), we finally obtain  $\|u_n - u\|_\gamma \rightarrow 0$ , and hence  $L$  is complete under  $\|\cdot\|_\gamma$ .

If  $\{u_n\} \subset L$  is a  $\|\cdot\|_\gamma$ -bounded sequence, then there are constants  $M_1, M_2, M_3$  such that

$$\int_0^\infty |u_n(t)|^2 dt \leq M_1, \quad \int_0^\infty |u'_n(t)|^2 dt \leq M_2, \quad \int_0^\infty p(t) |u_n(t)|^2 dt \leq M_3 \quad (27)$$

hold for all  $n \geq 1$ . By the relation  $u_n(x)^2 = 2 \int_0^x u_n(t) u'_n(t) dt$ , we have

$$|u_n(x)|^2 \leq 2 \int_0^\infty |u_n(t)| |u'_n(t)| dt \leq 2 \sqrt{\int_0^\infty |u_n(t)|^2 dt} \sqrt{\int_0^\infty |u'_n(t)|^2 dt} \leq \sqrt{M_1 M_2},$$

so that  $\{u_n(x)\}$  is uniformly bounded. Since for any  $0 \leq x < y$ ,

$$|u_n(x) - u_n(y)| \leq \int_x^y |u'_n(t)| dt \leq \sqrt{\int_x^y 1 dt} \sqrt{\int_0^\infty |u'_n(t)|^2 dt} \leq \sqrt{M_2} |y - x|$$

holds for all  $n \geq 1$ , the sequence  $\{u_n\}$  is uniformly equi-continuous. Hence it contains a subsequence  $\{u_{n'}(x)\}$  which converges uniformly on every compact subset of  $[0, \infty)$  to a continuous function  $u(t)$ . Now for sufficiently large  $X > 0$  it holds that

$$\int_X^\infty |u_n(t)|^2 dt \leq \frac{1}{\inf_{s \geq X} p(s)} \int_X^\infty p(t) |u_n(t)|^2 dt \leq \frac{M_3}{\inf_{s \geq X} p(s)},$$

and so

$$\begin{aligned} & \int_0^\infty |u_{n'}(t) - u(t)|^2 dt \\ & \leq \int_0^X |u_{n'}(t) - u(t)|^2 dt + \int_X^\infty |u(t)|^2 dt + \int_X^\infty |u_{n'}(t)|^2 dt \\ & \leq \int_0^X |u_{n'}(t) - u(t)|^2 dt + \liminf_{m' \rightarrow \infty} \int_X^\infty |u_{m'}(t)|^2 dt + \int_X^\infty |u_{n'}(t)|^2 dt \\ & \leq \int_0^X |u_{n'}(t) - u(t)|^2 dt + \frac{2M_3}{\inf_{s \geq X} p(s)}. \end{aligned}$$

By letting  $n' \rightarrow \infty$  first, and then  $X \rightarrow \infty$ , we obtain  $\lim_{n' \rightarrow \infty} \|u_{n'} - u\|_2 = 0$ .

We can now apply general theory, to obtain a self-adjoint operator  $A_\omega$  with domain  $D(A_\omega) \subset L$  such that  $(A_\omega u, v) = \mathcal{E}_\omega(u, v)$  for all  $u, v \in D(A_\omega)$ . In particular,  $\psi \in D(A_\omega)$  if and only if there is a  $\varphi \in L^2(\mathbf{R}_+)$  such that  $\mathcal{E}_\omega(u, \psi) = (u, \varphi)$  for all  $u \in L$  (see [12], Theorem VIII.15). Moreover, the assertion (iv) above implies that  $A_\omega$  has purely discrete spectrum (see [13], Theorem XIII.64). We shall show that  $A_\omega = H_0^S(\omega)$  holds for every fixed  $\omega$  for which the conclusion of Lemma 1 is valid. Let us begin by proving the following assertion.

**Proposition 2** *The relation  $H_0^S(\omega) \subset A_\omega \subset H_0(\omega)$  holds.*

**Proof.** We prove  $A_\omega \subset H_0(\omega)$  only. The other inclusion relation follows by taking the adjoint.

From the definition of the quadratic form  $\mathcal{E}_\omega$ , it is easy to see that  $\psi \in D(A_\omega)$  if and only if there is a  $\varphi \in L^2(\mathbf{R}_+)$  such that

$$\begin{aligned} & \int_0^\infty u'(t) \{ \overline{\psi'(t)} - c(X_\omega(t) - a_\omega(t)) \overline{\psi(t)} \} dt \\ &= \int_0^\infty u(t) \{ \overline{\varphi(t)} - (p(t) + ca'_\omega(t)) \overline{\psi(t)} + c(X_\omega(t) - a_\omega(t)) \overline{\psi'(t)} \} dt \end{aligned} \quad (28)$$

holds for all  $u \in L$ . In particular, if  $\psi \in D(A_\omega)$ , then the relation (28) holds for all  $u \in L$  with compact support. Then by Lemma 2 below, we can conclude that

$$\psi'(t) - cX_\omega(t) + ca_\omega(t)\psi(t) = \psi^{[1]}(t) + ca_\omega(t)\psi(t)$$

is absolutely continuous and

$$-(\psi^{[1]}(t) + ca_\omega(t)\psi(t))' = \varphi(t) - (p(t) + ca'_\omega(t))\psi(t) + c(X_\omega(t) - a_\omega(t))\psi'(t),$$

namely

$$\varphi(t) = -(\psi^{[1]}(t))' - cX_\omega(t)\psi^{[1]}(t) + (p(t) - c^2X_\omega(t)^2)\psi(t). \quad (29)$$

This shows that  $\psi \in \mathcal{D}_0(\omega)$  and  $H_0(\omega)\psi = \varphi$ .

**Lemma 2** *Let  $\alpha(t)$  be locally bounded and measurable,  $\gamma(t)$  be locally integrable on  $[0, \infty)$ . If*

$$\int_0^\infty u'(t)\alpha(t)dt = \int_0^\infty u(t)\gamma(t)dt \quad (30)$$

*holds for all  $u \in L$  with compact support, then  $\alpha(t)$  is absolutely continuous and  $-\alpha'(t) = \gamma(t)$  holds almost everywhere on  $[0, \infty)$ .*

**Proof.** For a given  $T > 0$ , it is easy to see that  $u \in L$  has its support in  $[0, T]$  if and only if there is a  $v \in L^2(\mathbf{R}_+)$  such that

$$u(t) = \int_0^{t \wedge T} (v(s) - \langle v \rangle_T) ds \quad (31)$$

holds, where we have set  $\langle v \rangle_T = \frac{1}{T} \int_0^T v(s) ds$ . Hence (30) becomes

$$\int_0^T (v(t) - \langle v \rangle_T) \alpha(t) dt = \int_0^T \left\{ \int_0^t (v(s) - \langle v \rangle_T) ds \right\} \gamma(t) dt,$$

which can be rewritten in the following form

$$\int_0^T v(t)(\alpha(t) - \langle \alpha \rangle_T) dt = \int_0^T v(t)(C(t) - \langle C \rangle_T) dt, \quad (32)$$



where  $C(t) := \int_t^T \gamma(s) ds$ . Since  $v$  runs through all of  $L^2([0, T])$ , we have  $\alpha(t) - \langle \alpha \rangle_T = C(t) - \langle C \rangle_T$  almost everywhere on  $[0, T]$ , so that  $\alpha'(t) = -\gamma(t)$ . Since  $T > 0$  is arbitrary, we arrive at the desired conclusion.

To the best of the author's knowledge, it was Fukushima and Nakao [5] who first treated one-dimensional Schrödinger operator with singular potential including Gaussian white noise. In fact, they considered the operator  $H_h := -d^2/dt^2 + h'(t)$  on a finite interval  $[a, b]$  under the Dirichlet boundary condition  $u(a) = u(b) = 0$ , where  $h(t)$  is a bounded Borel function on  $[a, b]$ . They defined it as the self-adjoint operator  $A_h$  associated to the quadratic form

$$\mathcal{E}_h(u, v) = \int_a^b u'(t) \overline{v(t)} dt - \int_a^b h(t) \{u(t) \overline{v(t)}\}' dt, \quad (33)$$

which is defined on

$$H_0^1(a, b) = \{u; u(\cdot) \text{ is absolutely continuous on } [a, b], u'(\cdot) \in L^2([a, b]), \text{ and } u(a) = u(b) = 0\} \quad (34)$$

and is closed, lower semi-bounded and completely continuous. In a similar manner as Lemma 2 above, it can be verified that a function  $\psi \in L^2([a, b])$  belongs to the domain of  $A_h$  if and only if  $\psi$  and  $\psi^{[1]} := \psi' - h\psi$  are absolutely continuous on  $[a, b]$ ,  $\psi(a) = \psi(b) = 0$  and  $\varphi := -(\psi^{[1]})' - h\psi^{[1]} - h^2\psi \in L^2([a, b])$ . And in this case, we have  $\varphi = A_h\psi$ . Thus the operator  $A_h$  of Fukushima and Nakao coincides with the operator described in §2.

If we introduce Prüfer variables  $r_\lambda(t)$  and  $\theta_\lambda(t)$  by

$$\psi(t) = r_\lambda(t) \sin \theta_\lambda(t), \quad \psi^{[1]}(t) = r_\lambda(t) \cos \theta_\lambda(t), \quad (35)$$

where  $\psi$  is a solution to  $H_h\psi = \lambda\psi$ , then in exactly the same way as the classical Sturm-Liouville theory, one can prove the following oscillation theorem.

**Proposition 3**  $\theta_\lambda(t)$  can be defined as a jointly continuous function of  $(t, \lambda)$ , which is strictly increasing in  $\lambda$  for each fixed  $t$ , and which satisfies  $d\theta_\lambda(t)/dt = 1$  whenever  $\theta_\lambda(t) \equiv 0 \pmod{\pi}$ . If  $\lambda_1 < \lambda_2 < \dots$  are the eigenvalues of  $A_h$ , then the eigenfunction  $\psi_h(t)$  belonging to the  $k$ -th eigenvalue  $\lambda_k$  has exactly  $k - 1$  zeros in  $(a, b)$ . In other words, there exist  $a < t_1 < \dots < t_{k-1} < b$  such that  $\theta_{\lambda_k}(t_j) = j\pi$  ( $j = 1, \dots, k - 1$ ).

Let us return to the proof of Theorem 2. Fix an  $\omega$  for which the conclusion of Lemma 1 is true, and let  $\lambda_1(\omega)$  and  $\psi_{1,\omega}(t)$  be the lowest eigenvalue of  $A_\omega$  and its eigenfunction. We shall show that  $\psi_{1,\omega}(t)$  has no zeros on  $(0, \infty)$ , and apply the following theorem of Hartman [6], to conclude that  $H(\omega)$  is of limit-point type at  $+\infty$  and that consequently  $H_0^S(\omega)$  is self-adjoint. We note that Hartman's theorem is valid also for our generalized Sturm-Liouville operator.

**Proposition 4** (Part (i) of Theorem in [6]) *Let  $H = H(p, Q)$  be a generalized Sturm-Liouville operator as in (9). If for some real number  $\mu$ , the equation  $H\varphi = \mu\varphi$  has a solution  $\psi$  with only finitely many zeros on  $[0, \infty)$ , then  $H$  is of limit-point type at  $+\infty$ .*

Let  $L_0$  be the subspace of  $L$  consisting of all compactly supported functions belonging to  $L$ . Then we have

$$\begin{aligned} \lambda_1(\omega) &= \inf\{\mathcal{E}_\omega(u, u); u \in L, \|u\|_2 = 1\} \\ &\leq \inf\{\mathcal{E}_\omega; u \in L_0, \|u\|_2 = 1\} \\ &= \lim_{l \rightarrow \infty} \inf\{\mathcal{E}_\omega; u \in L_0, \|u\|_2 = 1, \text{ supp } u \subset [0, l]\}. \end{aligned} \quad (36)$$

If we denote by  $\lambda_1^l(\omega)$  the infimum appearing on the right hand side of (36), then  $\lambda_1^l(\omega)$  is nothing but the lowest eigenvalue of  $H(\omega)$  on  $[0, l]$  considered under the Dirichlet boundary condition  $u(0) = u(l) = 0$ . Hence its eigenfunction  $\psi_{1,\omega}^l(t)$  has no zeros on  $(0, l)$ .

Now suppose  $\psi_{1,\omega}(t)$  has zeros on  $(0, \infty)$  and let  $l$  be the smallest among them, so that  $\psi_{1,\omega}(t) \neq 0$  for  $0 < t < l$  and  $\psi_{1,\omega}(l) = 0$ . This shows that the function  $\psi_{1,\omega}(t)$ , considered on  $0 \leq t \leq l$ , is the first eigenfunction of  $H(\omega)$  on  $[0, l]$  with the Dirichlet boundary condition at both endpoints, namely  $\lambda_1(\omega) = \lambda_1^l(\omega)$ . If  $\theta_\lambda(t)$ ,  $\lambda = \lambda_1(\omega) = \lambda_1^l(\omega)$ , is the Prüfer variable for  $\psi_{1,\omega}(t)$ , then we have  $\theta_\lambda(0) = 0$ ,  $0 < \theta_\lambda(t) < \pi$  for  $0 < t < l$ , and  $\theta_\lambda(l) = \pi$ . On the other hand, if we take  $l' > l$  and if we let  $\theta_{\lambda'}(t)$ ,  $\lambda' := \lambda_1^{l'}(\omega)$ , be the Prüfer variable for the first eigenfunction  $\psi_{1,\omega}^{l'}(t)$  of  $H(\omega)$  on  $[0, l']$  with the Dirichlet boundary condition at both endpoints, then we have  $\theta_{\lambda'}(0) = 0$ ,  $0 < \theta_{\lambda'}(t) < \pi$  for  $0 < t < l'$ , and  $\theta_{\lambda'}(l') = \pi$ . Comparing this behavior of  $\theta_{\lambda'}(t)$  to that of  $\theta_\lambda(t)$  above, we must conclude  $\lambda' < \lambda$  from the strict monotonicity of  $\theta_\lambda(t)$  with respect to  $\lambda$ , namely  $\lambda_1^{l'}(\omega) < \lambda_1^l(\omega) = \lambda_1(\omega)$  for any  $l' > l$ , contradicting (36). The proof of Theorem 2 is now complete.

**Remark 2** (i) For  $k \geq 2$ , let  $\lambda_k(\omega)$  and  $\psi_{k,\omega}(t)$  be the  $k$ -th eigenvalue and eigenfunction of  $A_\omega = H_0^S(\omega)$ , where the conclusion of Lemma 1 is valid for  $\omega$ . Then  $\psi_{k,\omega}(t)$  has exactly  $k - 1$  zeros on  $(0, \infty)$ . This can be proved in the same way as above if we note the following min-max characterization of  $\lambda_k(\omega)$ :

$$\lambda_k(\omega) = \inf_M \sup \{ \mathcal{E}_\omega(u, u); u \in M, \|u\|_2 = 1 \}, \quad (37)$$

where  $M$  runs through all  $k$ -dimensional linear subspaces of  $L$ .

(ii) Once we have verified that  $\psi_{k,\omega}(t)$  has only finitely many zeros for all  $k$ , it is now obvious that for any real  $\lambda$ , all non-trivial solution of  $H(\omega)u = \lambda u$  has only finitely many zeros on  $(0, \infty)$ .

Thus from Corollary in [6], we see that  $H(\omega)$ , considered under the boundary condition  $u(0) \cos \theta + u'(0) \sin \theta = 0$ , also defines a semi-bounded self-adjoint operator with purely discrete spectrum. The  $k$ -th eigenfunction of this operator has exactly  $k - 1$  zeros on  $(0, \infty)$ .

## 4 Remarks and applications.

### 4.1 Explosion of the solution of a stochastic Riccati equation.

Let  $H$  be a generalized Sturm-Liouville operator described in §2, and let  $u(t)$  be a non-trivial solution of the equation  $Hu = \lambda u$ . If we let  $z_\lambda(t) = u'(t)/u(t)$ , then it is easily seen that it satisfies

$$z_\lambda(t) - z_\lambda(s) = Q(t) - Q(s) + \int_s^t \{p(y) - \lambda - z_\lambda(y)^2\} dy \quad (38)$$

on every interval on which  $u(t)$  never vanishes, and that  $z_\lambda(\tau \pm 0) = \pm\infty$  whenever  $u(\tau) = 0$ .

If  $Q(t) = cX_\omega(t)$ ,  $X_\omega(t)$  being a sample function of the fractional Brownian motion of Hurst parameter  $h \in (0, 1)$ , then (38) defines a stochastic process  $\{z_{\lambda,\omega}(t)\}$  with state space  $[-\infty, +\infty]$ . If, in addition,  $p(t)$  satisfies the condition of Theorem 2, then from Remark 2, we obtain the following result.

**Proposition 5** For  $\mathbf{P}$ -almost all  $\omega \in \Omega$ , it holds that  $\{z_{\lambda,\omega}(t)\}$  explodes to  $-\infty$  finitely often for any real  $\lambda$  and for any initial value  $z(0)$  including  $+\infty$ .

### 4.2 Schrödinger operator with Gaussian white noise potential under a uniform electric field.

In [9], the present author considered a random Schrödinger operator on  $[0, \infty)$  of the form

$$H_{\kappa,F}^\theta(\omega) = -\frac{d^2}{dt^2} + \kappa B_\omega'(t) - Ft, \quad (39)$$

under the boundary condition  $u(0) \cos \theta + u'(0) \sin \theta = 0$  at the origin. Here  $\{B_\omega(t)\}$  is the standard Brownian motion as before and the constants  $\kappa$  and  $F$  are strictly positive. The following result was obtained.

**Theorem 3** ([9]) *For any fixed value of  $F > 0$ ,  $\kappa > 0$  and  $\theta \in [0, \pi)$ , the operator  $H_{\kappa, F}^\theta(\omega)$  is, with probability one, self-adjoint, its spectrum being  $(-\infty, \infty)$ . When  $0 < F < \kappa^2/2$ , the spectrum of  $H_{\kappa, F}^\theta(\omega)$  is almost surely pure point, while when  $F \geq \kappa^2/2$ , it is purely singular continuous.*

Combining Theorem 3, together with its proof in [9], and Theorem 2 of the present work, we obtain the following result.

**Theorem 4** *The random Schrödinger operator*

$$H_{\kappa, F}(\omega) = -\frac{d^2}{dt^2} + \kappa B'_\omega(t) - Ft$$

*on the whole space  $\mathbf{R}$  is almost surely a self-adjoint operator in the Hilbert space  $L^2(\mathbf{R})$ , and its spectrum as a set is  $(-\infty, \infty)$ .  $H_{\kappa, F}(\omega)$  has pure point spectrum with probability one when  $0 < F < \kappa^2/2$ , while it has purely singular continuous spectrum if  $F \geq \kappa^2/2$ .*

**Proof.** Theorems 2 and 3 tells us that  $H_{\kappa, F}(\omega)$  is almost surely of limit point type both at  $+\infty$  and  $-\infty$ , so that it defines a unique self-adjoint operator almost surely. We know that Condition K in [9] holds for  $H_{\kappa, F}(\omega)$ . Moreover it was proved in [9] that when  $0 < F < \kappa^2/2$ , the condition of Theorem 4 (ii) there is satisfied, while when  $F \geq \kappa^2/2$ , that of Theorem 4 (vi) holds. On the other hand, by Theorem 2, Remark 2 and Lemma in [6], we see that with probability one, the equation  $H_{\kappa, F}(\omega)u = \lambda u$  has a non-trivial solution which is square integrable near  $-\infty$  for every real  $\lambda$ . Hence accordingly as  $0 < F < \kappa^2/2$  or  $F \geq \kappa^2/2$ ,  $H_{\kappa, F}(\omega)$  satisfies the condition (i) or (v) of Theorem 4 of [9], and the desired conclusion follows.

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